Computational Advances for Cancer Detection Through Electrical Impedance Tomography and Optimal Control Theory

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Introduction

We propose a set of novel approaches combined in the fully developed computational framework **EIT-OPT** for the optimal reconstruction of **binary-type images** suitable for various models seen in biomedical applications. This framework enables accurate solutions to the inverse problem of cancer detection (**IPCD**) while applying the electrical impedance tomography (**EIT**) for detecting multiple cancer-affected regions of different sizes and different levels of complexity based on available (noisy) measurements. A new spatial partitioning methodology and efficient scheme for switching between fine and coarse scales allow higher variations in the geometry of reconstructed binary images with superior performance confirmed computationally on various models. The efficiency in computational speed and accuracy is achieved by combining the advantages of recently developed optimization methods that use sample solutions with customized geometry and control space reduction based on the samples' geometry and individual contributions paired with gradient-based techniques. A nominal number of input parameters makes the approaches simple for practical implementation in diverse settings and extendable to the broad range of problems in biomedical sciences, physics, geology, chemistry, etc.

IPCD: Mathematical Model

Forward problem: peripheral electrodes $(E_\ell)_{\ell=1}^m$ apply potentials $(U_\ell)_{\ell=1}^m$ and initiate currents $(I_\ell)_{\ell=1}^m$

$$\nabla \cdot [\sigma(x) \nabla u(x)] = 0, \qquad x \in \Omega \subset \mathbb{R}^n,\ n = 2,3$$

$$\frac{\partial u(x)}{\partial n} = 0, \qquad x \in \partial\Omega - \bigcup_{\ell=1}^{m} E_\ell$$

$$u(x) + Z_\ell \sigma(x) \frac{\partial u(x)}{\partial n} = U_\ell, \qquad x \in E_\ell$$

$$\int_{E_\ell} \sigma(x) \frac{\partial u(x)}{\partial n}\, ds = I_\ell, \qquad \ell = 1, \ldots, m$$

Optimization: solve with Km sets of data I_ℓ^{j*} [1]

$$\min_\sigma \mathcal{J}(\sigma) = \sum_{j=1}^{Km} \sum_{\ell=1}^{m} \beta_\ell^j \left[\int_{E_\ell} \frac{U_\ell^j - u^j(x;\sigma)}{Z_\ell}\, ds - I_\ell^{j*} \right]^2$$

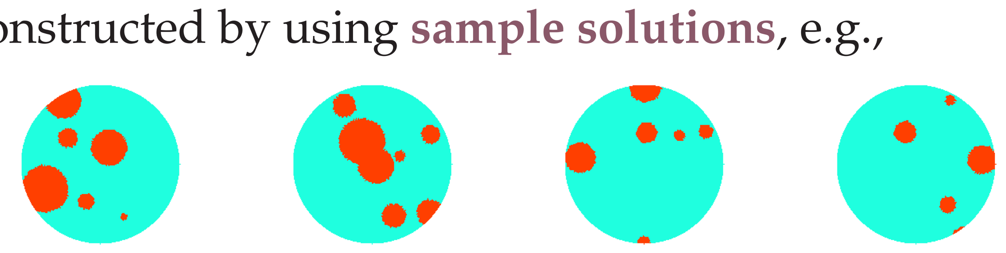
for electrical (isotropic) conductivity $\sigma(x) : \Omega \to \mathbb{R}_+$

Fine Scale Solution by PCA

PCA-based parameterization due to **ill-posedness**

$$\sigma = \Phi\,\xi + \bar{\sigma}, \quad \xi = \hat{\Phi}^{-1}(\sigma - \bar{\sigma})$$

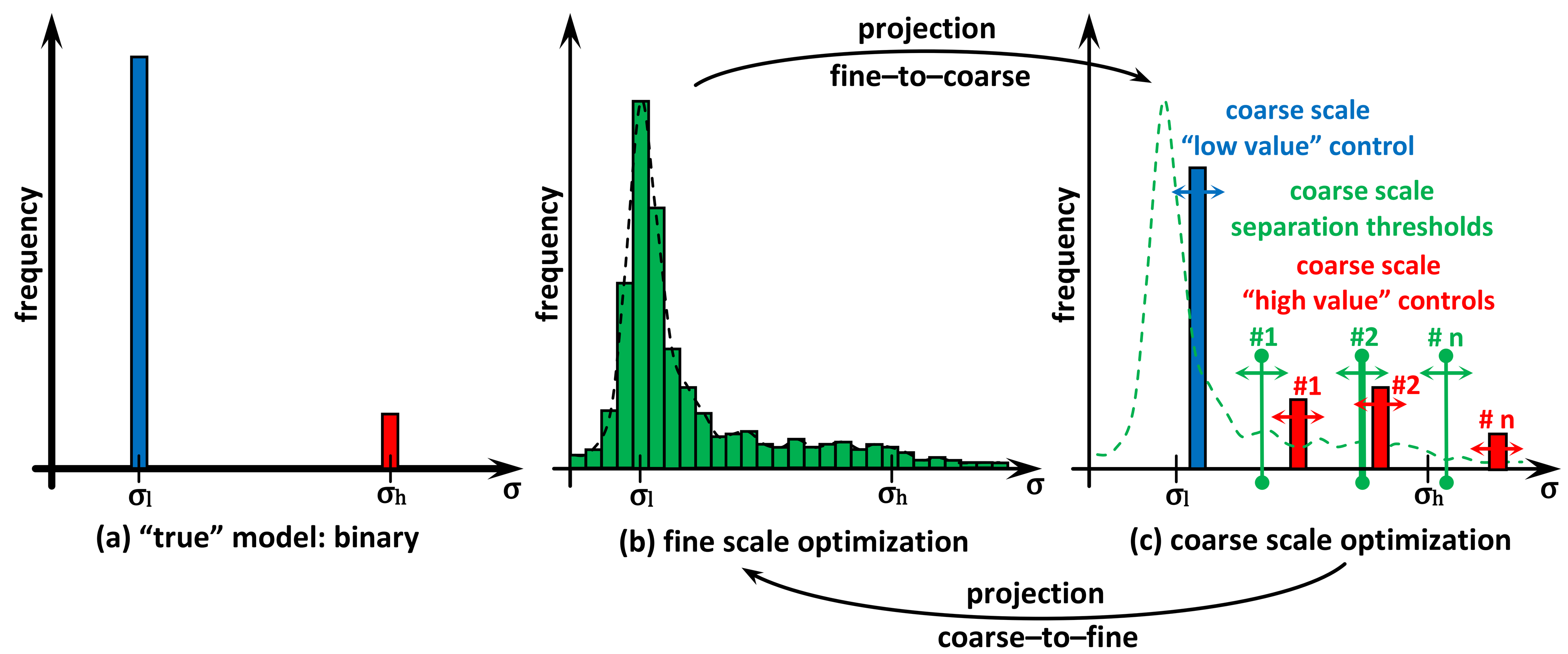
constructed by using **sample solutions**, e.g.,



and **restated optimization problem**

$$\hat{\xi} = \underset{\xi}{\operatorname{argmin}}\ \mathcal{J}(\xi(\sigma))$$

Multilevel Multiscale Control Space Reduction



Gradient-based optimization at various levels (scales) [1, 2]:

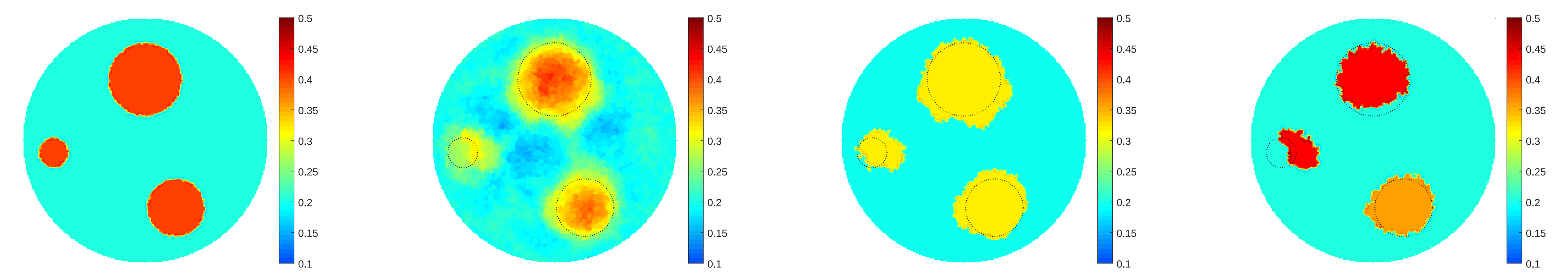
- **adjoint** (ψ) gradients: $\nabla_\sigma \mathcal{J} = -\sum_{j=1}^{Km} \nabla \psi^j(x) \cdot \nabla u^j(x)$
- **fine-scale** (PCA-projected to ξ-space): $\nabla_\xi \mathcal{J} = \Phi^T \nabla_\sigma \mathcal{J}$
- **coarse-scale:** $\nabla_\zeta \mathcal{J} : \frac{\partial \mathcal{J}}{\partial \zeta_j} = \sum_{i=1}^{N} P_{i,j} \frac{\partial \mathcal{J}}{\partial \sigma_i} \Delta_i$ upscaled with **fine–to-coarse partitioning** $\mathcal{M}$ (N fine vs. N_ζ coarse elements)

$$\mathcal{M} : (\sigma_i)_{i=1}^N \to \bigcup_{j=1}^{N_\zeta} C_j, \quad C_j = \{\sigma_i : P_{i,j} = 1,\ i = 1, \ldots, N\},$$

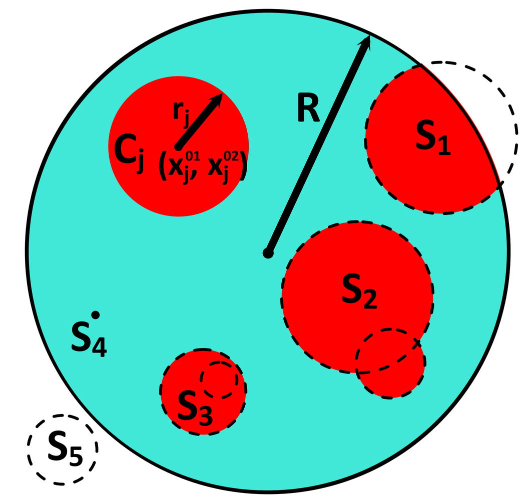
$$\sum_{j=1}^{N_\zeta} |C_j| = \sum_{j=1}^{N_\zeta} N_j = N, \quad P_{i,j} = \begin{cases} 1, & \sigma_i \in C_j, \\ 0, & \sigma_i \notin C_j. \end{cases}$$

Results: Model #1 (left to right)

- **true** conductivity $\sigma_{true}(x)$
- optimal $\hat{\sigma}$ with **fine scale** only
- $\hat{\sigma}$: $N_\zeta = 2$ (**"one"** spot)
- $\hat{\sigma}$: $N_\zeta = 6$ (**multiple** spots)



Sample-based Parameterization



Model #1 & **Model #2**
(top & bottom rows)

Solution: electrical conductivity $\sigma(x)$ by N_s samples [3, 4]

$$\sigma(x) = \sum_{i=1}^{N_s} \alpha_i \bar{\sigma}_i(x), \quad 0 \le \alpha_i \le 1, \quad \sum_{i=1}^{N_s} \alpha_i = 1$$

from **N-sample collection** (generated randomly, $N \gg N_s$)

$$\mathcal{C}(N) = (\bar{\sigma}_i(x))_{i=1}^N, \quad \bar{\sigma}_i(x) = \begin{cases} \sigma_c, & |x - x_j^0|^2 \le r_j^2,\ j = 1, \ldots, N_c^i \\ \sigma_h, & \text{otherwise} \end{cases}$$

Parameterization: N_c^i circles of radii r and centers (x^{01}, x^{02})

$$\mathcal{P}_i = (\{x_j^{01}, x_j^{02}, r_j\})_{j=1}^{N_c^i}, \quad i = 1, \ldots, N$$

STEP 1: initial basis $\mathcal{B}^0 = (\bar{\sigma}_i(x))_{i=1}^{N_s}$ by best samples out of $\mathcal{C}(N)$

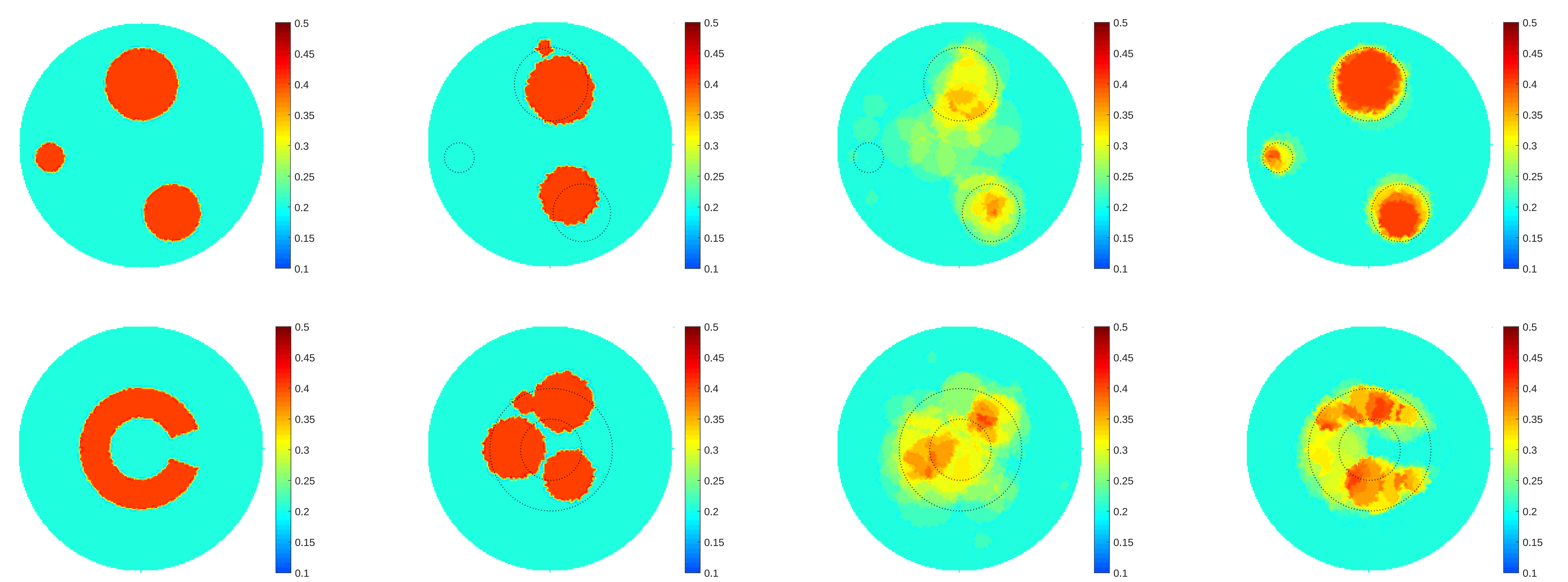
STEP 2: find **optimal** basis $\hat{\mathcal{B}}$ by solving optimal control problem

$$(\hat{\mathcal{P}}, \hat{\alpha}) = \underset{\mathcal{P},\alpha}{\operatorname{argmin}}\ \mathcal{J}(\mathcal{P}, \alpha)$$

Results: (left to right)

- **true** conductivity $\sigma_{true}(x)$
- **best sample** $\bar{\sigma}_1(x)$ from initial basis $\mathcal{B}^0$
- **STEP 1:** complete **initial basis** $\mathcal{B}^0$ approximation $\sigma^0(x)$
- **STEP 2: optimal basis** $\hat{\mathcal{B}}$ approximation $\hat{\sigma}(x)$

Parameters: $N = 10{,}000$, $N_s = 8$
Optimization: customized coordinate descent method [3, 4]



References

[1] **Koolman, P. M. & Bukshtynov, V.** *A Multiscale Optimization Framework for Reconstructing Binary Images using Multilevel PCA-based Control Space Reduction*, Biomed. Phys. Eng. Express 7(2) (2021), 025005.

[2] **Chun, M. M. F. M.** *Multiscale Optimization via Multilevel PCA-based Control Space Reduction in Applications to Electrical Impedance Tomography*, M.S. Thesis, (2022).

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