Gradient Method in Besov Spaces for the Optimal Control of Parabolic Free Boundary Problems

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Abstract

We present computational analysis of the inverse Stefan type free boundary problem, where information on the boundary heat flux is missing and must be found along with the temperature and the free boundary. We pursue optimal control framework introduced in *U.G. Abdulla, Inverse Problems and Imaging, 7, 2(2013), 307-340; 10, 4(2016), 869–898*, where boundary heat flux and free boundary are components of the control vector, and optimality criteria consist of the minimization of the quadratic declinations from the available measurements of the temperature distribution at the final moment, phase transition temperature on the free boundary, and the final position of the free boundary. We develop gradient descent algorithm based on Fréchet differentiability in Hilbert-Besov spaces complemented with preconditioning or increase of regularity of the Fréchet gradient through implementation of the Riesz representation theorem. Three model examples with various levels of complexity are considered. Extensive comparative analysis through implementation of preconditioning and Tikhonov regularization, calibration of preconditioning and regularization parameters, effect of noisy data, comparison of simultaneous identification of control parameters vs. nested optimization is pursued.

Optimal Control Problem

Consider a minimization of the cost functional

$$\mathcal{J}(v) = \beta_0 \|u(x,T) - w(x)\|^2_{L_2[0,s(T)]} + \beta_1 \|u(s(t),t) - \mu(t)\|^2_{L_2[0,T]} + \beta_2 |s(T) - s_*|^2$$

on the control set

$$V_R = \{v = (s,g) \in B_2^2[0,T] \times B_2^1[0,T] : \delta \le s(t) \le l, s(0) = s_0, s'(0) = 0, \max(\|s\|_{B_2^2};\ \|g\|_{B_2^1} \le R\}$$

where $\delta, l, R, \beta_0, \beta_1$ are given positive numbers, and $u = u(x,t;v)$ be a solution of the Neumann problem.

Fréchet Differentiability

Theorem: The functional $\mathcal{J}(v)$ is differentiable in the sense of Fréchet, and the Fréchet differential is

$$\langle \mathcal{J}'(v), \delta v \rangle_H = -\int_0^T \psi(0,t)\delta g(t)\,dt - \int_0^T \left[\gamma\psi\right]_{x=s(t)} \delta s'(t)\,dt + \int_0^T \Big[2\beta_1(u-\mu)u_x + \psi\left(\chi_x - \gamma_x s' - \left(a u_x\right)_x\right)\Big]_{x=s(t)} \delta s(t)\,dt + \Big(\beta_0 u(s(T),T) - w(s(T))^2 + 2\beta_2(s(T) - s_*)\Big)\delta s(T),$$

where $\mathcal{J}'(v) \in H'$ is the Fréchet derivative, $\langle \cdot,\cdot \rangle_H$ is a pairing between H and its dual H', ψ is a solution to the adjoint problem, and $\delta v = (\delta s, \delta g)$ is a variation of the control vector $v \in V_R^1$ such that $v + \delta v \in V_R^1$.

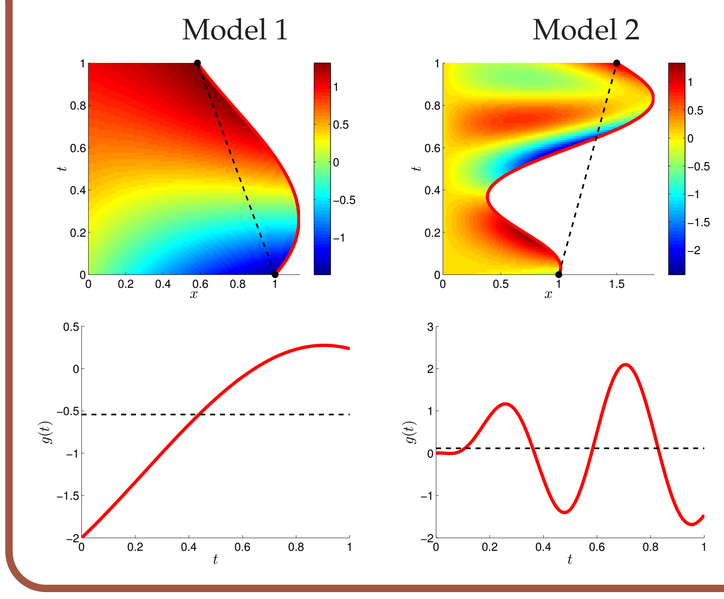
Statement of Problem

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t)$$
where $(x,t) \in \Omega$
$$u(x,0) = \phi(x),\ 0 \le x \le s(0) = s_0$$
$$a(0,t)u_x(0,t) = g(t),\ 0 \le t \le T$$
$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t),$$
where $0 \le t \le T$
$$u(s(t),t) = \mu(t),\ 0 \le t \le T,$$

and

$$\Omega = \{(x,t) : 0 < x < s(t),\ 0 < t \le T\}$$

Computational Models



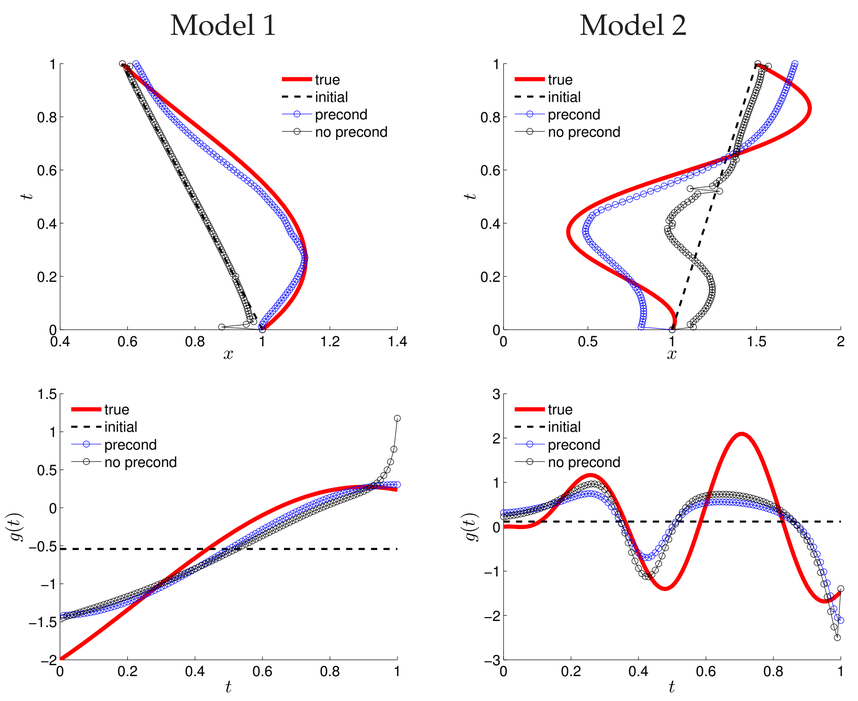
Numerical Results

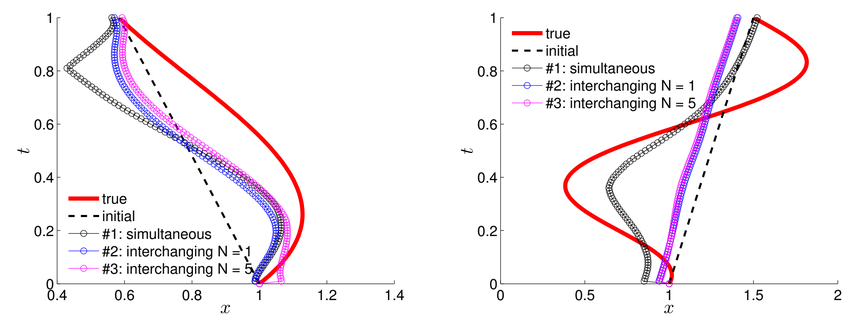
Numerical algorithm

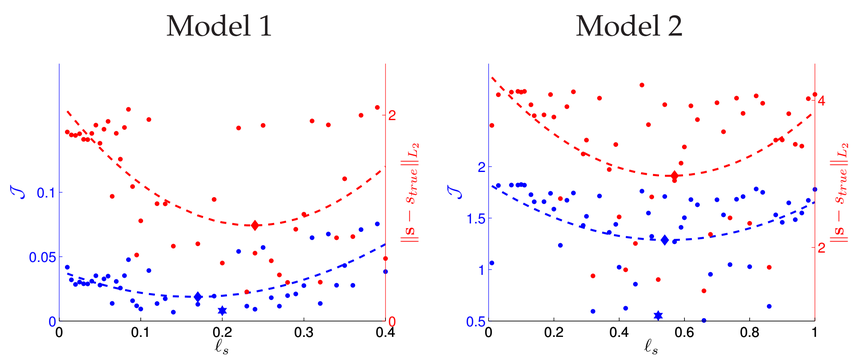
- **Step 1.** Set $k = 0$ and choose initial vector function $v_0 = (s_0, g_0) \in V_R$.
- **Step 2.** Solve the Neumann problem (above) to find $u_k = u(x,t;v_k)$ and (v_k).
- **Step 3.** If $k = 0$, move to Step 4. Otherwise, check the following criteria:

$$\left|\frac{\mathcal{J}(v_k) - \mathcal{J}(v_{k-1})}{\mathcal{J}(v_{k-1})}\right| < \epsilon, \qquad \frac{v_k - v_{k-1}}{v_{k-1}} < \epsilon,$$

  where ϵ is the required accuracy. If the criteria are satisfied, then terminate the iteration. Otherwise, move to Step 4.
- **Step 4.** Having u_k, solve the adjoined PDE problem to find $\psi_k = \psi(x,t;v_k)$.
- **Step 5.** Choose stepsize parameter $\alpha_k > 0$ and compute new control vector $v_{k+1} = (s_{k+1}, g_{k+1}) \in H$
- **Step 6.** Replace v_{k+1} with $\mathcal{P}_{V_R}(v_{k+1})$, where $\mathcal{P}_{V_R} : H \to V_R^1$ is projection operator to closed and convex subset V_R^1. Then replace k with $k+1$ and move to Step 2.







Projecting gradients via preconditioning

Solve the following BVP to perform Step 6:

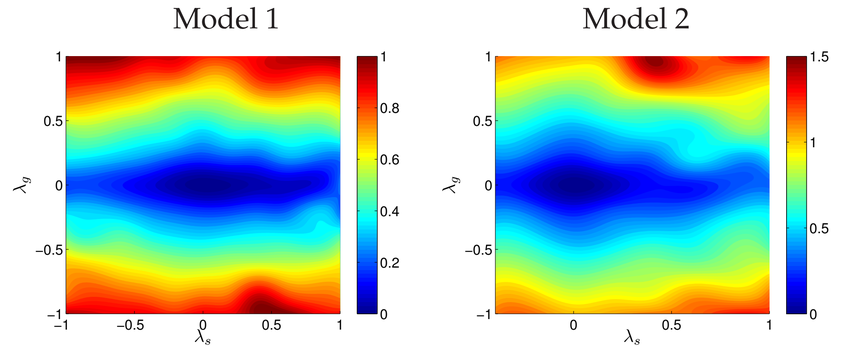
$$\nabla_s^{H^1}\mathcal{J} - \ell_s^2 \frac{d^2}{dt^2}\nabla_s^{H^1}\mathcal{J} = \nabla_s^{L_2}\mathcal{J} \quad \text{on } (0,T)$$
$$\frac{d}{dt}\nabla_s^{H^1}\mathcal{J} = 0 \quad \text{for } t = 0,\ T.$$

Sensitivity analysis w.r.t. initial guess

Initial guesses $s_{\mathrm{ini},\lambda_s}$, $g_{\mathrm{ini},\lambda_g}$ are parameterized

$$s_{\mathrm{ini},\lambda_s}(t) = (1-\lambda_s)s_{\mathrm{true}}(t) + \lambda_s s_{\mathrm{ini}}(t),$$
$$g_{\mathrm{ini},\lambda_g}(t) = (1-\lambda_g)g_{\mathrm{true}}(t) + \lambda_g g_{\mathrm{ini}}(t)$$

with respect to their proximity to global minimizers s_{true} and g_{true} respectively.



References

1. U.G. Abdulla, On the optimal control of the free boundary problems for the second order parabolic equations. I. Well-posedness and convergence of the method of lines, Inverse Problems and Imaging 7(2) (2013), 307–340.
2. U.G. Abdulla, V. Bukshtynov, A. Hagverdiyev, Gradient Method in Hilbert–Besov Spaces for the Optimal Control of Parabolic Free Boundary Problems. Journal of Computational and Applied Mathematics 346 (2019), 84–109.